MATH3280A Introductory Probability, 2014-2015
Solutions to HW4

P.182 Ex.5

Solution

$$\begin{aligned}
E(X) &= \sum_{k=1}^{N} k \cdot P(X=k) \\
&= \sum_{k=1}^{N} k \cdot \frac{1}{N} \\
&= \frac{N(N+1)}{2} \cdot \frac{1}{N} \\
&= \frac{N+1}{2}
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \sum_{k=1}^{N} k^2 \cdot \frac{1}{N} \\
&= \frac{N(N+1)(2N+1)}{6} \cdot \frac{1}{N} \\
&= \frac{(N+1)(2N+1)}{6}
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E(X^2) - E(X)^2 \\
&= \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4} \\
&= \frac{N+1}{12}[2(2N+1) - 3(N+1)] \\
&= \frac{N^2-1}{12}
\end{aligned}$$

$$\sigma_X = \sqrt{\frac{N^2-1}{12}}$$

□

P.212 Ex.10

Solution
Let X be the random variable of the number of members who were born on Independence Day. We have $X \sim b(26, \frac{1}{365})$.

(a) Using binomial distribution

$$p_i = P(X = i) = \binom{26}{i} \left(\frac{1}{365}\right)^i \left(\frac{364}{365}\right)^{26-i}$$

$$p_0 \approx 0.93115$$

$$p_1 \approx 0.06651$$

$$p_2 \approx 0.00228$$

$$p_3 \approx 0.00005$$

(b) Using Poisson distribution

We approximate X by Poisson random variable $Y \sim po(\lambda)$, where $\lambda = \frac{26}{365}$.

$$p_i \approx P(Y = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

$$p_0 \approx 0.93125$$

$$p_1 \approx 0.06634$$

$$p_2 \approx 0.00236$$

$$p_3 \approx 0.00006$$

$\square$

P.258 Ex.4

Solution

$$
\begin{aligned}
P(-2 < X < 1) &= \int_{-2}^{1} f(x)dx \\
&= \int_{-2}^{1} \frac{e^{-|x|}}{2} dx \\
&= \frac{1}{2}\left(\int_{-2}^{0} e^{x} dx + \int_{0}^{1} e^{-x} dx\right) \\
&= \frac{1}{2}\left((1 - e^{-2}) + (-e^{-1} + 1)\right) \\
&= \frac{2 - e^{-1} - e^{-2}}{2}
\end{aligned}
$$

□

P.258 Ex.6

Solution
The density function of X is

$$f_X(x) = \begin{cases} \frac{4x^3}{15} & \text{, if } 1 \le x \le 2 \\ 0 & \text{, otherwise} \end{cases}$$

Then the distribution function of X is

$$F_X(t) = \begin{cases} 0 & \text{, if } t < 1 \\ \int_1^t \frac{4x^3}{15} dx & \text{, if } 1 \le t \le 2 \\ 1 & \text{, if } t > 2 \end{cases}$$

The distribution function of $Y = exp(X)$ is

$$\begin{aligned} F_Y(t) &= P(exp(X) \le t) \\ &= \begin{cases} 0 & \text{, if } t \le 0 \\ F_X(ln(t)) & \text{, if } t > 0 \end{cases} \\ &= \begin{cases} 0 & \text{, if } t \le 0 \text{ or } ln(t) < 1 \\ F_X(ln(t)) & \text{, if } 1 \le ln(t) \le 2 \\ 1 & \text{, if } ln(t) > 2 \end{cases} \\ &= \begin{cases} 0 & \text{, if } t < e \\ F_X(ln(t)) & \text{, if } e \le t \le e^2 \\ 1 & \text{, if } t > e^2 \end{cases} \end{aligned}$$

Suppose Y is a continuous random variable with density function f_Y.
Then $f_Y = F'_Y$ almost everywhere.

$$\begin{aligned} F'_Y(t) &= \begin{cases} 0 & \text{, if } t < e \text{ or } t > e^2 \\ F'_X(ln(t)) \cdot \frac{1}{t} & \text{, if } e < t < e^2 \end{cases} \\ &= \begin{cases} 0 & \text{, if } t < e \text{ or } t > e^2 \\ \frac{4(ln(t))^3}{15t} & \text{, if } e < t < e^2 \end{cases} \end{aligned}$$

Define

$$f_Y(t) = \begin{cases} 0 & \text{, if } t < e \text{ or } t > e^2 \\ \dfrac{4(ln(t))^3}{15t} & \text{, if } e \leq t \leq e^2 \end{cases}$$

Then we can check that

$$\int_{-\infty}^{t} f_Y(x)dx = F_Y(t), \text{ for any } t \in \mathbb{R}.$$

Hence f_Y is a density function for Y.

The distribution function of $Z = X^2$ is

$$\begin{aligned}
F_Z(t) &= P(X^2 \le t) \\
&= \begin{cases} 0 & \text{, if } t \le 0 \\ P(-\sqrt{t} \le X \le \sqrt{t}) & \text{, if } t > 0 \end{cases} \\
&= \begin{cases} 0 & \text{, if } t \le 0 \\ F_X(\sqrt{t}) - F_X(-\sqrt{t}) & \text{, if } t > 0 \text{ (Note: } F_X(-\sqrt{t}) = 0) \end{cases} \\
&= \begin{cases} 0 & \text{, if } t \le 0 \text{ or } \sqrt{t} < 1 \\ F_X(\sqrt{t}) & \text{, if } 1 \le \sqrt{t} \le 2 \\ 1 & \text{, if } \sqrt{t} > 2 \end{cases} \\
&= \begin{cases} 0 & \text{, if } t < 1 \\ F_X(\sqrt{t}) & \text{, if } 1 \le t \le 4 \\ 1 & \text{, if } t > 4 \end{cases}
\end{aligned}$$

We have

$$F'_Z(t) = \begin{cases} 0 & \text{, if } t < 1 \text{ or } t > 4 \\ \dfrac{4(\sqrt{t})^3}{15} \dfrac{1}{2\sqrt{t}} & \text{, if } 1 < t < 4 \end{cases}$$

Define

$$f_Z(t) = \begin{cases} 0 & \text{, if } t < 1 \text{ or } t > 4 \\ \dfrac{2t}{15} & \text{, if } 1 \le t \le 4 \end{cases}$$

Then we can check that

$$\int_{-\infty}^{t} f_Z(x)dx = F_Z(t), \text{ for any } t \in \mathbb{R}.$$

Hence f_Z is a density function for Z.

The distribution function of $W = (X-1)^2$ is

$$\begin{aligned}
F_W(t) &= P((X-1)^2 \le t) \\
&= \begin{cases} 0 & \text{, if } t < 0 \\ P(-\sqrt{t} \le X - 1 \le \sqrt{t}) & \text{, if } t \ge 0 \end{cases} \\
&= \begin{cases} 0 & \text{, if } t < 0 \\ F_X(1+\sqrt{t}) - F_X(1-\sqrt{t}) & \text{, if } t \ge 0 \end{cases} \\
&= \begin{cases} 0 & \text{, if } t < 0 \\ F_X(1+\sqrt{t}) & \text{, if } 1 \le 1+\sqrt{t} \le 2 \\ 1 & \text{, if } 1+\sqrt{t} > 2 \end{cases} \\
&= \begin{cases} 0 & \text{, if } t < 0 \\ F_X(1+\sqrt{t}) & \text{, if } 0 \le t \le 1 \\ 1 & \text{, if } t > 1 \end{cases}
\end{aligned}$$

We have

$$F'_W(t) = \begin{cases} 0 & \text{, if } t < 0 \text{ or } t > 1 \\ \dfrac{4(1+\sqrt{t})^3}{15} \dfrac{1}{2\sqrt{t}} & \text{, if } 0 < t < 1 \end{cases}$$

Define

$$f_W(t) = \begin{cases} 0 & \text{, if } t < 0 \text{ or } t > 1 \\ \dfrac{2(1+\sqrt{t})^3}{15\sqrt{t}} & \text{, if } 0 \le t \le 1 \end{cases}$$

Then we can check that

$$\int_{-\infty}^{t} f_W(x)dx = F_W(t), \text{ for any } t \in \mathbb{R}.$$

Hence f_W is a density function for W.

$\square$